Global Existence of a Virus Infection Model with Saturated Chemotaxis

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Abstract: In this paper, a virus infection model with saturated chemotaxis is formulated and analyzed, where the chemotactic sensitivity for chemotactic movements of the cells is described. This model contains three state variables namely the population density of uninfected cells, the population density of infected cells and the concentration of virus particles, respectively. By virtue of regularized approximation technique and fixed point theorem, the local solvability of the regularized system corresponding to the original system is established. Then by extracting a suitable sequence along which the respective approximate solutions approach a limit in convenient topologies, with addition of Gagliardo-Nirenberg interpolation inequality as well as L^p-estimate techniques, we show that the original system describing the virus infection model exists at least one global weak solution. To illustrate the application of our theoretical results, an optimal control problem of the epidemic system is considered, where the admissible control domain is assumed to be a bounded closed convex subset. With the help of Aubin compactness theorem and lower semicontinuous of the cost functional, the existence of the optimal pair is proved. Our results generalize and improve partial previously known ones, and moreover, we first prove that the optimal control problem has at least one optimal pair.

Keywords: Virus Infection, Global Existence, Chemotaxis, Optimal Control

1. Introduction

As the virological, immunological and mathematical plates become interlocked, mathematics is playing an increasingly important role in biology. Many mathematicians began to use the rigorous theory and methods of partial differential equations to elaborate and forecast some complex biological phenomena. Especially for the virus infection dynamic models with diffusion terms. The vast majority of existing research on evolution of a virus infection model was almost described by ordinary differential equations (ODEs) [7, 18], this leads to the ignorance of spatial variations, which means the ODEs are not suitable for obtaining spatial information about the distribution of infected cells. To make up for this deficiency, spatial dependence of the virus infection dynamic models must be taken into consideration [5].

It is well known that some epidemic diseases are awful if no effective measures are taken to control them, such as cholera, tuberculosis and so on. Thus, the optimal control problems of epidemic models have been drawing more and more notice in recent decades. For example, Kirschner et al. studied optimal chemotherapy strategy in an early treatment background which depicted the interaction of the immune system with the human immunodeficiency virus (HIV) by the optimal control theories and methods, where the immune system is governed by ODEs [13]. Chang and Astolfi used the drug scheduling methods to measure the states of the HIV model on the basis of a reduced-order model framework, and presented the corresponding simulation results [4]. Also, with the help of optimal control methods, Xiang and Liu solved the inverse problem of an SIS epidemic model of the ecosystem [34, 35]. Meantime, Zhou et al. used the two control treatment, that is, vaccination and therapy to consider the optimal control problem of an epidemic system governed by reaction-diffusion equations [36].

Recent modeling methods and experimental results show that the chemotactic sensitivity is in general a tensor for chemotactic movements of the cells [33]. In this paper, we concern with the following virus infection model with saturated chemotaxis:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (uS(x,u,v)\nabla v) - uw + \kappa - u, & x \in \Omega, \ t > 0, \\ v_t = \Delta v + uw - v, & x \in \Omega, \ t > 0, \\ w_t = \Delta w + v - w, & x \in \Omega, \ t > 0, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. u, v and w denote the population density of uninfected cells, the population density of infected cells and the concentration of virus particles, respectively. Initial data u_0, v_0, w_0 are known functions satisfying

$$u_0 \in C^0(\overline{\Omega}), \ v_0 \in W^{1,\infty}(\Omega), \ w_0 \in C^0(\overline{\Omega}),$$
$$u_0 \geq 0, \ v_0 \geq 0, \ w_0 \geq 0. \quad (2)$$

We suppose that $S \in C^2(\overline{\Omega} \times [0,\infty)^2; \mathbb{R}^{N \times N})$ denoting the rotational effect, which is induced by a swimming bias and the bacteria themselves, has the property that there exist $S_0 > 0$ and $\alpha > 0$ fulfilling

$$|S(x,u,v)| \leq S_0 \cdot (1+u)^{-\alpha}$$
$$\text{for all } x \in \overline{\Omega}, u \geq 0 \text{ and } v \geq 0. \quad (3)$$

Then we will consider (1) along with the initial conditions:

$$u(x,0) = u_0(x), \ v(x,0) = v_0(x),$$
$$w(x,0) = w_0(x), \quad x \in \Omega \quad (4)$$

and the boundary conditions:

$$(\nabla u - uS(x,u,v) \cdot \nabla v) \cdot \nu = 0,$$
$$\frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, \ x \in \partial\Omega, \ t > 0. \quad (5)$$

In fact, the dynamics of high HIV seminal loads leading to sporadic infection are difficult to understand biologically and completely, when they are falling outside the scope of usual mathematical modelling of infectious diseases described by simple ODEs. In order to better understand the formation of patterns on the onset of an HIV infection, Stancevic et al. proposed the following mathematical model [26]

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + \kappa - (\kappa-1)uw - u, \\ v_t = D_1\Delta v + \alpha(uw - v), \\ w_t = D_2\Delta w + \beta(v - w), \end{cases} \quad (6)$$

where u, v and w denote the population density of uninfected cells, the population density of infected cells and the concentration of virus particles, respectively. $\chi, \kappa, \alpha, \beta, D_1, D_2$ are suitable positive constants. The virus is also produced by infected cells and its presence causes healthy cells to be converted into infected cells. Furthermore, healthy cells are produced with a constant rate κ. $\chi\nabla \cdot (u\nabla v)$ describes chemotactic response to cytokines emitted by infected cells moving toward high concentration. However, the pioneering work of the chemotaxis model was first introduced by Keller and Segel [12], where aggregation of cellular slime mold toward a higher concentration of a chemical signal was described by:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), \\ v_t = \Delta v + u - v, \end{cases} \quad (7)$$

where u denotes the cell density and v is the chemical concentration. The mathematical analysis of (7) and the variant thereof mainly concentrated on the boundedness and blow-up of the solutions [6, 10]. In addition to the original model, a large number of variants of the classical form have also been studied, including the systems with the logistic terms [19], chemotaxis-haptotaxis models [20], multi-species chemotaxis systems [1, 21, 24], attraction-repulsion chemotaxis system [23, 25], chemotaxis-fluid model [14, 22, 30] and so on. During the past four decades, the chemotaxis model has become one of the best study models in mathematical biology. And we refer the reader to the survey [3, 8, 9], in which we can find further examples to illustrate the significant biological correlation of chemotaxis.

It is well known that the cross-diffusive term in (6) is the key contributor to analyze the global existence in mathematics. In order to exclude the possibility of blow-up, motivated by [26, Sec. 8], Hu and Lankeit considered the following system [11]

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\frac{u}{(1+u)^\alpha}\nabla v) - uw + \kappa - u, \\ v_t = \Delta v + uw - v, \\ w_t = \Delta w + v - w, \end{cases} \quad (8)$$

where $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$ is a bounded domain with smooth boundary $\partial\Omega$ and $\frac{\partial}{\partial\nu}$ denotes the derivative with respect to the outer normal of $\partial\Omega$. In which, they proved that if

$$\begin{cases} \alpha > \frac{2}{3}, & \text{if } N = 1, \\ \alpha > \frac{1}{2} + \frac{N^2}{6N+4}, & \text{if } 2 \leq N \leq 4, \\ \alpha > \frac{N}{4}, & \text{if } N \geq 5, \end{cases}$$

hold, then the system (8) existed a global bounded solution. For a related system, such as HBV infection model, see also [28, 29].

To the best of our knowledge, the optimal control problem of virus infection models with saturated chemotaxis (1) has not been studied. With the addition of the arguments in previous studies [11, 16, 17, 22, 24, 30, 33], the aim of this paper is to consider a virus infection model with saturated chemotaxis. Under appropriate regularity assumptions on the initial data, via L^p-estimate techniques, we show that the epidemic system (1) exists at least one global weak solution. This result generalizes and improves Theorem 1.1 [11]. Moreover, the existence of the optimal pair of system (1) is obtained.

In this paper, we use symbols C_i and c_i $(i = 1, 2, \cdots)$ as some generic positive constants which may vary from line to line. For simplicity, $u(x, t)$ is written as u, the integral $\int_\Omega u(x)dx$ is written as $\int_\Omega u(x)$ and $\int_0^t \int_\Omega u(x)dxdt$ is written as $\int_0^t \int_\Omega u(x)$.

The contents of the paper are as follows. In Section 2, some basic definitions and main theorems as well as some useful lemmas are presented. In Section 3, some fundamental estimates for the solution of the system (14) are given, and the previously mentioned a priori estimate in the process of limit procedure is discussed and Theorem 2.1 is proved. In Section 4, the optimal control problem of the system (70) is considered and the existence of the optimal pair is obtained.

2. Preliminaries

In order to consider the optimal control of virus infection model with saturated chemotaxis, it is necessary to first discuss the well-posedness of the system (1). Inspired by [33], see also [30], the concept of the solution is presented.

Definition 2.1. Assume that S complies with (3), $\Phi \in C^2([0, \infty))$ be a nonnegative function fulfilling $\Phi' > 0$ on $(0, \infty)$. Suppose that $u_0 \in L^\infty(\Omega)$ is nonnegative and that $\Phi(u_0) \in L^1(\Omega)$. Furthermore, let $v \in L^2_{loc}([0, \infty); W^{1,2}(\Omega)) \cap L^\infty_{loc}(\overline{\Omega} \times [0, \infty))$ and $w \in L^2_{loc}([0, \infty); W^{1,2}(\Omega))$. The nonnegative measurable function $u : \Omega \times (0, \infty) \to \mathbb{R}$ will be named a global weak Φ-supersoluton of the initial boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (uS(x, u, v)\nabla v) - uw + \kappa - u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (9)$$

if

$$\Phi(u), \Phi''(u)|\nabla u|^2, \Phi'(u)uw, \Phi'(u) \in L^1_{loc}(\overline{\Omega} \times [0, \infty)), u\Phi''(u)\nabla u, \Phi'(u)u \in L^2_{loc}(\overline{\Omega} \times [0, \infty)),$$

and if for each nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))$ with $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega \times (0, \infty)$, the inequality

$$-\int_0^\infty \int_\Omega \Phi(u)\varphi_t - \int_\Omega \Phi(u_0)\varphi(\cdot, 0) \geq \int_0^\infty \int_\Omega \Phi(u)\Delta\varphi$$

$$-\int_0^\infty \int_\Omega \Phi''(u)|\nabla u|^2\varphi \int_0^\infty - \int_0^\infty \int_\Omega \Phi'(u)uw\varphi$$

$$+\int_0^\infty \int_\Omega u\Phi''(u)\nabla u \cdot (S(x, u, v) \cdot \nabla v)\varphi + \kappa \int_0^\infty \int_\Omega \Phi'(u)\varphi$$

$$+\int_0^\infty \int_\Omega u\Phi'(u)(S(x, u, v) \cdot \nabla v) \cdot \nabla\varphi - \int_0^\infty \int_\Omega u\Phi'(u)\varphi$$

is satisfied.

Definition 2.2. A triplet (u, v, w) of functions

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0, \infty)), \\ v \in L^2_{loc}([0, \infty); W^{1,2}(\Omega)), \\ w \in L^2_{loc}([0, \infty); W^{1,2}(\Omega)), \end{cases}$$

fulfilling $u \geq 0$, $v \geq 0$ and $w \geq 0$ in $\overline{\Omega} \times [0, \infty)$, $uw \in L^1_{loc}(\overline{\Omega} \times [0, \infty))$ will be called a global weak solution of (1), (4) and (5), if

$$\int_\Omega u(\cdot, t) \leq e^{-t} \int_\Omega u_0 + \kappa|\Omega|(1 - e^{-t}) \quad for \ a.e. \ t > 0, \quad (10)$$

and if the equality

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0\varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega uw\varphi - \int_0^\infty \int_\Omega v\varphi \quad (11)$$

as well as

$$-\int_0^\infty \int_\Omega w\varphi_t - \int_\Omega w_0\varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla w \cdot \nabla\varphi + \int_0^\infty \int_\Omega v\varphi - \int_0^\infty \int_\Omega w\varphi \quad (12)$$

hold for all $\varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ with $\varphi_t \in L^2(\Omega \times (0,\infty))$, which are compactly supported in $\overline{\Omega} \times [0,\infty)$, and if finally there exists some nonnegative function $\Phi \in C^2([0,\infty))$ with $\Phi' > 0$ on $(0,\infty)$ such that u is a global weak Φ-supersoluton of (9) in the sense of Definition 2.1.

Remark 2.1. Following the proof of a demonstration [33, Lemma 2.1], together with (10), we know that if global weak solution (u,v,w) fulfills $u,v,w \geq 0$ and the regularity properties $u,v,w \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty))$, then (u,v,w) is also a classical solution of (1) in $\Omega \times (0,\infty)$.

Then, we state our main result as follows.

Theorem 2.1. Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary. Assume that S complies with (3). If $\alpha > \frac{N+1}{N+2}$, then for any choice of the initial data (u_0, v_0, w_0) satisfy (4), system (1) admits at least one global weak solution in the sense of Definition 2.2.

Remark 2.2. When $N = 1, 2$, Theorem 2.1 is consistent with the results of [11, Theorem 1.1], when $N \geq 3$, Theorem 2.1 generalizes and improves the results of [11, Theorem 1.1].

The proof of Theorem 2.1, we left it in Section 3.

In order to construct such weak solutions by an approximation procedure, we fix $(\rho_\varepsilon)_{\varepsilon \in (0,1)} \subset C_0^\infty([0,\infty))$ and $(\chi_\varepsilon)_{\varepsilon \in (0,1)} \subset C^\infty([0,\infty))$ be cut-off functions such that

$$0 \leq \rho_\varepsilon \leq 1 \text{ in } \Omega \text{ with } \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon \searrow 0,$$

and that

$$0 \leq \chi_\varepsilon \leq 1 \text{ in } [0,\infty) \text{ with } \chi_\varepsilon \equiv 0 \text{ in } [\frac{1}{\varepsilon}, \infty)$$

and

$$\chi_\varepsilon \nearrow 1 \text{ in } [0,\infty) \text{ as } \varepsilon \searrow 0.$$

For $\varepsilon \in (0,1)$, we then define

$$S_\varepsilon(x,u,v) := \rho_\varepsilon(x) \cdot \chi_\varepsilon(u) \cdot S(x,u,v),$$
$$(x,u,v) \in \overline{\Omega} \times [0,\infty)^2. \tag{13}$$

We will construct solutions of (1) as limits of solutions to relevant regularized approximate problems, and give some basic estimates for the solutions to the regularized system. For any such ε, the regularized problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon) - u_\varepsilon w_\varepsilon + \kappa - u_\varepsilon, & x \in \Omega, \; t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon + u_\varepsilon w_\varepsilon - v_\varepsilon, & x \in \Omega, \; t > 0, \\ w_{\varepsilon t} = \Delta w_\varepsilon + v_\varepsilon - w_\varepsilon, & x \in \Omega, \; t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = \frac{\partial w_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \; t > 0, \\ u_\varepsilon(x,0) = u_0(x), \; v_\varepsilon(x,0) = v_0(x), \; w_\varepsilon(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{14}$$

According to the well-established fixed point arguments, the local solvability of (14) can be obtained, the proof is similar to [10, 32], so here we omit the proof.

Lemma 2.1. Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$ be a bounded domain with smooth boundary. Assume that S complies with (3), (u_0, v_0, w_0) satisfy (4). Suppose that $\alpha, \kappa \geq 0$. Then for each $\varepsilon \in (0,1)$, there exist $T_{\max} \in (0,\infty]$ and a classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ such that

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ v_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ w_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})). \end{cases}$$

where $T_{\max}$ denotes the maximal existence time. Moreover, we have $u_\varepsilon \geq 0$, $v_\varepsilon \geq 0$ and $w_\varepsilon \geq 0$ in $\overline{\Omega} \times (0, T_{\max})$, and if $T_{\max} < +\infty$, then

$$\lim_{t \to T_{\max}} \Big(\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{15}$$

Then, some important and useful lemmas are presented to prove the main theorems.

Lemma 2.2. Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary. Assume that S complies with (1.3), (n_0, c_0, v_0, u_0) satisfy (4). Suppose that $\alpha, \kappa \geq 0$. Then for each $\varepsilon \in (0,1)$,

$$\int_\Omega u_\varepsilon(\cdot,t) \leq C_1 := e^{-t} \int_\Omega u_0 + \kappa|\Omega|(1 - e^{-t}) \text{for all } t \in (0, T_{\max}) \tag{16}$$

Proof. Integrating the first equation of (14) and using the nonnegativity of $u_\varepsilon w_\varepsilon$, we can immediately derive (16). This completes the proof.

Lemma 2.3. Let the assumptions in Lemma 2.2 hold. Then for each $\varepsilon \in (0,1)$,

$$\int_\Omega (u_\varepsilon(\cdot,t) + v_\varepsilon(\cdot,t)) \leq C_2 := e^{-t} \int_\Omega (u_0 + v_0) + \kappa|\Omega|(1 - e^{-t}) \quad \text{for all } t \in (0, T_{\max}). \tag{17}$$

In particular, we have

$$\int_\Omega v_\varepsilon(\cdot,t) \leq C_2 \quad \text{for all } t \in (0, T_{\max}). \tag{18}$$

Proof. The proof is similar to [11, Lemma 2.2] and [11, Corollary 2.3], so we omit it.

Lemma 2.4. For each $\varepsilon \in (0,1)$, the solution of (14) is global-in-time; that is, we have $T_{\max} = \infty$ in Lemma 2.1.

Proof. The proof is similar to our recent work [22, Lemma 3.2], so we omit it.

Lemma 2.5. (Gagliardo-Nirenberg interpolation inequality) ([15, Lemma 2.4]) Let $0 < \theta \leq p \leq \frac{2N}{N-2}$. There exists a positive constant C_{GN} such that for all $u_\varepsilon \in W^{1,2}(\Omega) \cap L^\theta(\Omega)$,

$$\|u_\varepsilon\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla u_\varepsilon\|_{L^2(\Omega)}^a \|u_\varepsilon\|_{L^\theta(\Omega)}^{1-a} + \|u_\varepsilon\|_{L^\theta(\Omega)})$$

is valid with $a = \frac{\frac{N}{\theta} - \frac{N}{p}}{1 - \frac{N}{2} + \frac{N}{\theta}} \in (0,1)$.

3. Proof of Theorem 2.1

3.1. A Priori Estimate

Lemma 3.1. Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary. Assume that S complies with (3). If $\alpha > \frac{N+1}{N+2}$, then for any choice of the initial data (u_0, v_0, w_0) satisfy (4), there exists $C > 0$ such that for all $\varepsilon \in (0,1)$

$$\int_t^{t+1} \int_\Omega |\nabla u_\varepsilon^\alpha|^2 \leq C \quad \text{for all } t \geq 0 \tag{19}$$

$$\int_t^{t+1} \int_\Omega |\nabla v_\varepsilon|^2 \leq C,$$

$$\int_t^{t+1} \int_\Omega |\nabla w_\varepsilon|^2 \leq C \quad \text{for all } t \geq 0 \tag{20}$$

$$\int_\Omega u_\varepsilon^{2\alpha}(\cdot, t) \leq C \quad \text{for all } t \geq 0 \tag{21}$$

and

$$\int_\Omega v_\varepsilon^2(\cdot, t) \leq C, \quad \int_\Omega w_\varepsilon^2(\cdot, t) \leq C \quad \text{for all } t \geq 0. \tag{22}$$

Proof. The strong maximum principle shows that u_ε is positive in $\overline{\Omega} \times (0, \infty)$. Multiplying the first equation in (14) by $u_\varepsilon^{2\alpha-1}$ and integrating by parts, using Young's inequality and (3), we have

$$\frac{1}{2\alpha} \frac{d}{dt} \int_\Omega u_\varepsilon^{2\alpha} + (2\alpha-1) \int_\Omega u_\varepsilon^{2\alpha-2} |\nabla u_\varepsilon|^2$$

$$\leq (2\alpha-1) \int_\Omega u_\varepsilon^{2\alpha-1} \nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon) - \int_\Omega u_\varepsilon^{2\alpha} w_\varepsilon + \kappa \int_\Omega u_\varepsilon^{2\alpha-1} - \int_\Omega u_\varepsilon^{2\alpha}$$

$$\leq \frac{2\alpha-1}{2} \int_\Omega u_\varepsilon^{2\alpha-2} |\nabla u_\varepsilon|^2 + \frac{2\alpha-1}{2} \int_\Omega u_\varepsilon^{2\alpha} |S_\varepsilon(x, u_\varepsilon, v_\varepsilon)|^2 |\nabla v_\varepsilon|^2 + \frac{2\alpha}{2\alpha-1} \int_\Omega u_\varepsilon^{2\alpha} + \frac{\kappa^{2\alpha}|\Omega|}{2\alpha} - \int_\Omega u_\varepsilon^{2\alpha} \tag{23}$$

$$\leq \frac{2\alpha-1}{2} \int_\Omega u_\varepsilon^{2\alpha-2} |\nabla u_\varepsilon|^2 + c_1 \int_\Omega |\nabla v_\varepsilon|^2 + \frac{1}{2\alpha-1} \int_\Omega u_\varepsilon^{2\alpha} + c_2$$

with $c_1 := \frac{2\alpha-1}{2} S_0^2$ and $c_2 := \frac{\kappa^{2\alpha}|\Omega|}{2\alpha}$. Since $\alpha > \frac{N+1}{N+2}$, we readily conclude that $\frac{2\alpha}{2\alpha-1} \in (1, 2 + \frac{2}{N})$, analogous to [11, Lemma 3.3], we obtain

$$\int_\Omega v_\varepsilon^{\frac{2\alpha}{2\alpha-1}} \leq \varepsilon_1 \int_\Omega |\nabla v_\varepsilon|^2 + C_1(\varepsilon_1) \quad \text{for all } t \geq 0 \tag{24}$$

for any $\varepsilon_1 > 0$, where $C_1(\varepsilon_1) > 0$ is a constant. We test the second equation in (14) by v_ε and integrate by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega v_\varepsilon^2 = \int_\Omega u_\varepsilon v_\varepsilon w_\varepsilon \quad \text{for all } t \geq 0. \tag{25}$$

By the Hölder inequality and Young's inequality, we obtain

$$\int_\Omega u_\varepsilon v_\varepsilon w_\varepsilon \leq \|w_\varepsilon\|_{L^\infty(\Omega)} \|u_\varepsilon\|_{L^{2\alpha}(\Omega)} \|v_\varepsilon\|_{L^{\frac{2\alpha}{2\alpha-1}}(\Omega)}$$

$$\leq c_3 \int_\Omega u_\varepsilon^{2\alpha} + (2\alpha-1) c_3 \int_\Omega v_\varepsilon^{\frac{2\alpha}{2\alpha-1}} \tag{26}$$

$$\leq c_3 \int_\Omega u_\varepsilon^{2\alpha} + (2\alpha-1) c_3 \varepsilon_1 \int_\Omega |\nabla v_\varepsilon|^2 + (2\alpha-1) c_3 C_1(\varepsilon_1),$$

where $c_3 > 0$ is a constant. Letting $\varepsilon_1 = \frac{1}{(2\alpha-1)c_3}$, substituting (26) into (25), we have

$$\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 + 2\int_\Omega v_\varepsilon^2 \leq 2c_3\int_\Omega u_\varepsilon^{2\alpha} + c_4 \quad \text{for all } t \geq 0. \tag{27}$$

with $c_4 = 2(2\alpha - 1)c_3 C_1(\varepsilon_1)$. Multiplying the third equation in (14) by w_ε and integrating by parts, using Young's inequality, we obtain that

$$\frac{d}{dt}\int_\Omega w_\varepsilon^2 + 2\int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega w_\varepsilon^2 \leq \int_\Omega v_\varepsilon^2 \quad \text{for all } t \geq 0. \tag{28}$$

Finally, taking an appropriate linear combination of above inequality with (23) and (27), we have

$$\frac{d}{dt}\left\{\frac{1}{2\alpha}\int_\Omega u_\varepsilon^{2\alpha} + 2c_1\int_\Omega v_\varepsilon^2 + c_1\int_\Omega w_\varepsilon^2\right\} + \frac{2\alpha-1}{2\alpha^2}\int_\Omega |\nabla u_\varepsilon^\alpha|^2 + c_1\int_\Omega |\nabla v_\varepsilon|^2$$
$$+ 3c_1\int_\Omega v_\varepsilon^2 + 2c_1\int_\Omega |\nabla w_\varepsilon|^2 + c_1\int_\Omega w_\varepsilon^2 \leq \left(\frac{1}{2\alpha-1} + 4c_1c_3\right)\int_\Omega u_\varepsilon^{2\alpha} + 2c_1c_4 + c_2. \tag{29}$$

By the Gagliardo-Nirenberg inequality and Young's inequality, there exists $c_5 > 0$ such that

$$\left(\frac{1}{2\alpha-1} + 4c_1c_3\right)\int_\Omega u_\varepsilon^{2\alpha} \leq \frac{2\alpha-1}{8\alpha^2}\int_\Omega |\nabla u_\varepsilon^\alpha|^2 + c_5 \quad \text{for all } t \geq 0. \tag{30}$$

Let

$$y(t) := \frac{1}{2\alpha}\int_\Omega u_\varepsilon^{2\alpha} + 2c_1\int_\Omega v_\varepsilon^2 + c_1\int_\Omega w_\varepsilon^2,$$
$$h(t) := \frac{2\alpha-1}{4\alpha^2}\int_\Omega |\nabla u_\varepsilon^\alpha|^2 + c_1\int_\Omega |\nabla v_\varepsilon|^2 + c_1\int_\Omega |\nabla w_\varepsilon|^2,$$

then since $\frac{2\alpha-1}{8\alpha^2}\int_\Omega |\nabla u_\varepsilon^\alpha|^2 + 3c_1\int_\Omega v_\varepsilon^2 + c_1\int_\Omega w_\varepsilon^2 \geq y(t)$ for all $t \geq 0$, from (29) and (30), we have

$$y'(t) + y(t) + h(t) \leq c_6 \quad \text{for all } t \geq 0 \tag{31}$$

with $c_6 := 2c_1c_4 + c_2 + c_5$, it immediately derives that

$$y(t) \leq c_7 := \max\left\{\frac{1}{2\alpha}\int_\Omega u_0^{2\alpha} + 2c_1\int_\Omega v_0^2 + c_1\int_\Omega w_0^2, \ c_6\right\} \quad \text{for all } t \geq 0. \tag{32}$$

Integrating (31) with respect to time, it immediately yields (19) and (20). This completes the proof.

Lemma 3.2. Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary. Assume that S complies with (3). Let $T > 0$, then there exists $C(T) > 0$ such that for each $\varepsilon \in (0, 1)$, the solution of (14) satisfies

$$\int_0^T \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \leq C(T). \tag{33}$$

Proof. Multiplying the first equation in (14) by $\frac{1}{u_\varepsilon+1}$, we have

$$\frac{d}{dt}\int_\Omega \ln(u_\varepsilon + 1) = -\int_\Omega \nabla u_\varepsilon \cdot \nabla \frac{1}{u_\varepsilon + 1} + \int_\Omega \nabla \frac{1}{u_\varepsilon + 1} \cdot (u_\varepsilon S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon)$$
$$- \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}w_\varepsilon + \kappa\int_\Omega \frac{1}{u_\varepsilon + 1} - \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}$$
$$= \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} - \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)^2} \cdot \nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \tag{34}$$
$$- \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}w_\varepsilon + \kappa\int_\Omega \frac{1}{u_\varepsilon + 1} - \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}$$

for all $t \geq 0$. By the Young's inequality and (3), we obtain

$$\left|-\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)^2} \cdot \nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon)\right| \leq \frac{1}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} + \frac{S_0^2}{2}\int_\Omega |\nabla v_\varepsilon|^2. \tag{35}$$

Thus, integrating (34) with respect to time, we have

$$\int_{\Omega} \ln(u_\varepsilon(\cdot, t) + 1) - \int_{\Omega} \ln(u_0 + 1) \geq \frac{1}{2} \int_0^t \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} - \frac{S_0^2}{2} \int_0^t \int_{\Omega} |\nabla v_\varepsilon|^2 - \int_0^t \int_{\Omega} w_\varepsilon(\cdot, t) - T|\Omega|. \tag{36}$$

By the variation-of-constants formula with respect to w_ε, we obtain

$$w_\varepsilon(\cdot, t) = e^{t(\Delta - 1)} w(\cdot, 0) + \int_0^t e^{(t-s)(\Delta - 1)} v(\cdot, s) ds,$$

from the Neumann heat semigroup theory [31, Lemma 1.3(i)], there exists $c_1, c_2 > 0$ such that

$$\|w_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq c_1 e^{-t} \|w_0\|_{L^1(\Omega)} + \int_0^t e^{-(t-s)} c_1 \|v_\varepsilon(\cdot, t)\|_{L^1(\Omega)} ds \leq c_2 \tag{37}$$

for all $t \in (0, T)$. Combining (36) with (37), we have

$$\frac{1}{2} \int_0^T \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \leq \int_{\Omega} u_\varepsilon(\cdot, t) + \frac{S_0^2}{2} \int_0^t \int_{\Omega} |\nabla v_\varepsilon|^2 + c_2 T + T|\Omega|$$

for all $t \in (0, T)$, here we have been used the fact that $0 \leq \ln(u_\varepsilon(\cdot, t) + 1) \leq u_\varepsilon(\cdot, t)$ for all $u_\varepsilon(\cdot, t) \geq 0$. By Lemma 3.1, the desired result is obtained. This completes the proof.

Lemma 3.3. Let $m \in \mathbb{N}$ be such that $m > \frac{N}{2}$. Then there exists $C > 0$ with the property that for each $\varepsilon \in (0, 1)$, the solution of (14) fulfills

$$\int_0^T \|\partial_t \ln(u_\varepsilon(\cdot, t) + 1)\|_{(W_0^{m,2}(\Omega))^*} dt \leq C(T + 1) \quad \text{for all } T > 0. \tag{38}$$

Proof. For fixed $t > 0$ and arbitrary $\phi \in W_0^{m,2}(\Omega)$, due to Sobolev embedding theorem, we see that $W_0^{m,2}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$. Multiplying the first equation in (14) by $\frac{1}{u_\varepsilon + 1} \phi$ and integrating by parts, we obtain that

$$
\begin{aligned}
\int_{\Omega} \partial_t \ln(u_\varepsilon + 1) \cdot \phi &= \int_{\Omega} \frac{1}{u_\varepsilon + 1} \Delta u_\varepsilon \cdot \phi - \int_{\Omega} \frac{1}{u_\varepsilon + 1} \nabla \cdot (u_\varepsilon S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon) \phi \\
&\quad - \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} w_\varepsilon \cdot \phi + \kappa \int_{\Omega} \frac{1}{u_\varepsilon + 1} \phi - \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} \phi \\
&= - \int_{\Omega} \frac{1}{u_\varepsilon + 1} \nabla u_\varepsilon \cdot \nabla \phi + \int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \cdot \phi - \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} w_\varepsilon \cdot \phi \\
&\quad - \int_{\Omega} \frac{u_\varepsilon}{(u_\varepsilon + 1)^2} \cdot \nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon) \phi - \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} \phi \\
&\quad + \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} (S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon) \cdot \nabla \phi + \kappa \int_{\Omega} \frac{1}{u_\varepsilon + 1} \phi.
\end{aligned}
\tag{39}
$$

Then, by the Cauchy-Schwarz inequality, we have

$$\left| - \int_{\Omega} \frac{1}{u_\varepsilon + 1} \nabla u_\varepsilon \cdot \nabla \phi \right| \leq \left(\int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \right)^{\frac{1}{2}} \cdot \|\nabla \phi\|_{L^2(\Omega)}, \tag{40}$$

$$\left| \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} (S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon) \cdot \nabla \phi \right| \leq S_0 \left(\int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\frac{1}{2}} \cdot \|\nabla \phi\|_{L^2(\Omega)}, \tag{41}$$

$$\left| - \int_{\Omega} \frac{u_\varepsilon}{(u_\varepsilon + 1)^2} \cdot \nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \nabla v_\varepsilon) \phi \right| \leq S_0 \left(\int_{\Omega} \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \right)^{\frac{1}{2}} \cdot \left(\int_{\Omega} |\nabla v_\varepsilon|^2 \right)^{\frac{1}{2}} \cdot \|\phi\|_{L^\infty(\Omega)}, \tag{42}$$

$$\left| - \int_{\Omega} \frac{u_\varepsilon}{u_\varepsilon + 1} w_\varepsilon \cdot \phi \right| \leq \|w_\varepsilon\|_{L^1(\Omega)} \cdot \|\phi\|_{L^\infty(\Omega)}, \tag{43}$$

$$\left| \kappa \int_{\Omega} \frac{1}{u_\varepsilon + 1} \phi \right| \leq \kappa |\Omega| \cdot \|\phi\|_{L^\infty(\Omega)} \tag{44}$$

as well as

$$\left|-\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\phi\right| \le |\Omega| \cdot \|\phi\|_{L^\infty(\Omega)}, \tag{45}$$

$$\left|\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\cdot\phi\right| \le \left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\right)\cdot\|\phi\|_{L^\infty(\Omega)}. \tag{46}$$

Substituting (40)-(46) into (39) and using Young's inequality, this implies that with some $c_1 > 0$,

$$\left|\int_\Omega \partial_t \ln(u_\varepsilon+1)\cdot\phi\right| \le c_1\Big\{1+\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\int_\Omega|\nabla v_\varepsilon|^2+\int_\Omega w_\varepsilon\Big\}\cdot\|\phi\|_{W_0^{m,2}(\Omega)}$$

for all $\phi \in W_0^{m,2}(\Omega)$, meaning that

$$\|\partial_t\ln(u_\varepsilon(\cdot,t)+1)\|_{(W_0^{m,2}(\Omega))^*} \le c_1\Big\{1+\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\int_\Omega|\nabla v_\varepsilon|^2+\int_\Omega w_\varepsilon\Big\}$$

for all $t > 0$. From Lemmas 3.1 and 3.2, we readily conclude that (38) by an integration over $(0,T)$. This completes the proof.

Now a straightforward application of the Aubin-Lions lemma can establish the following compactness properties of $(\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)}$.

Corollary 3.1. Let $T > 0$. Then $(\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)}$ is relatively compact in $L^2((0,T);W^{1,2}(\Omega))$ with respect to the weak topology, and relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology.

Proof. The proof is similar to [30, Corollary 5.3] and [33, Corollary 4.3], so we omit it.

Lemma 3.4. Let $T > 0$. Then $\{v_\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology.

Proof. For fixed $t > 0$ and arbitrary $\phi \in W_0^{m,2}(\Omega)$, due to Sobolev embedding theorem, we see that $W_0^{m,2}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$. Then from the second equation in (14) and the Cauchy-Schwarz inequality, Poincaré inequality, for each $t \in (0,T)$,

$$\begin{aligned}
\left|\int_\Omega v_{\varepsilon t}\phi\right| &= \left|-\int_\Omega \nabla v_\varepsilon\cdot\nabla\phi+\int_\Omega u_\varepsilon w_\varepsilon\phi-\int_\Omega v_\varepsilon\phi\right|\\
&\le \left(\int_\Omega|\nabla v_\varepsilon|^2\right)^{\frac{1}{2}}\cdot\|\nabla\phi\|_{L^2(\Omega)}+\|u_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|w_\varepsilon\|_{L^6(\Omega)}\|\phi\|_{L^\infty(\Omega)}\\
&\quad +\|v_\varepsilon\|_{L^1(\Omega)}\|\phi\|_{L^\infty(\Omega)}\\
&\le \left(\int_\Omega|\nabla v_\varepsilon|^2\right)^{\frac{1}{2}}\cdot\|\nabla\phi\|_{L^2(\Omega)}+c_1\|u_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}\|\nabla w_\varepsilon\|_{L^2(\Omega)}\|\phi\|_{L^\infty(\Omega)}\\
&\quad +\|v_\varepsilon\|_{L^1(\Omega)}\|\phi\|_{L^\infty(\Omega)}
\end{aligned}$$

with some certain constant $c_1 > 0$. By the Young's inequality, there exists $c_2 > 0$ such that

$$\int_0^T \|v_{\varepsilon t}\|_{(W_0^{m,2}(\Omega))^*}dt \le c_2\int_0^T\Big\{1+\int_\Omega|\nabla v_\varepsilon|^2+\int_\Omega v_\varepsilon+\|u_\varepsilon\|_{L^{\frac{6}{5}}(\Omega)}^2+\|\nabla w_\varepsilon\|_{L^2(\Omega)}^2\Big\}dt.$$

Hence, in accordance with Lemmas 2.3 and 3.1, we conclude that

$$\int_0^T \|v_{\varepsilon t}\|_{(W_0^{m,2}(\Omega))^*}dt \le c_3(T+1)$$

with constant $c_3 > 0$. Therefore, the Aubin-Lions lemma [27, Lemma 2.3] along with the boundedness of $(v_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2((0,T);W^{1,2}(\Omega))$ yields the claim. This completes the proof.

Lemma 3.5. Let $T > 0$. Then $\{w_\varepsilon\}_{\varepsilon\in(0,1)}$ is relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology.

Proof. For fixed $t > 0$ and arbitrary $\phi \in W_0^{m,2}(\Omega)$, due to Sobolev embedding theorem, we see that $W_0^{m,2}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$. Then from the third equation in (14) and the Cauchy-Schwarz inequality, for each $t \in (0,T)$,

$$\begin{aligned}
\left|\int_\Omega w_{\varepsilon t}\phi\right| &= \left|-\int_\Omega \nabla w_\varepsilon\cdot\nabla\phi-\int_\Omega w_\varepsilon\phi+\int_\Omega v_\varepsilon\phi\right|\\
&\le \left(\int_\Omega|\nabla w_\varepsilon|^2\right)^{\frac{1}{2}}\cdot\|\nabla\phi\|_{L^2(\Omega)}+\|w_\varepsilon\|_{L^1(\Omega)}\|\phi\|_{L^\infty(\Omega)}+\|v_\varepsilon\|_{L^1(\Omega)}\|\phi\|_{L^\infty(\Omega)}.
\end{aligned}$$

By the young's inequality, there exists $c_1 > 0$ such that

$$\int_0^T \|w_{\varepsilon t}\|_{(W_0^{m,2}(\Omega))^*} dt \leq c_1 \int_0^T \left\{ 1 + \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega v_\varepsilon + \int_\Omega w_\varepsilon \right\} dt.$$

Hence, in accordance with Lemmas 2.3 and 3.1, we conclude that

$$\int_0^T \|w_{\varepsilon t}\|_{(W_0^{m,2}(\Omega))^*} dt \leq c_2(T+1)$$

with constant $c_2 > 0$. Therefore, the Aubin-Lions lemma [27, Lemma 2.3] along with the boundedness of $(w_\varepsilon)_{\varepsilon \in (0,1)}$ in $L^2((0,T); W^{1,2}(\Omega))$ yields the claim. This completes the proof.

3.2. Passing to the Limit

Now, we are capable of extracting a suitable sequence of ε along which the respective solutions approach a limit in convenient topologies.

Lemma 3.6. Let $\alpha > \frac{N+1}{N+2}$ and assume that (u_0, v_0, w_0) satisfy (4). Then there exist $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and functions

$$u_\varepsilon \to u \quad a.e. \text{ in } \Omega \times (0, \infty), \quad (47)$$

$$\ln(u_\varepsilon + 1) \rightharpoonup \ln(u+1) \text{ in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \quad (48)$$

and

$$v_\varepsilon \to v \quad \text{in } L^2_{loc}(\bar{\Omega} \times [0, \infty)), \quad (49)$$

$$v_\varepsilon \to v \quad a.e. \text{ in } \Omega \times (0, \infty), \quad (50)$$

$$\nabla v_\varepsilon \rightharpoonup \nabla v \quad \text{in } L^2_{loc}(\bar{\Omega} \times [0, \infty)), \quad (51)$$

as well as

$$w_\varepsilon \to w \quad \text{in } L^2_{loc}(\bar{\Omega} \times [0, \infty)), \quad (52)$$

$$w_\varepsilon \to w \quad a.e. \text{ in } \Omega \times (0, \infty), \quad (53)$$

$$\nabla w_\varepsilon \rightharpoonup \nabla w \quad \text{in } L^2_{loc}(\bar{\Omega} \times [0, \infty)), \quad (54)$$

$$u_\varepsilon w_\varepsilon \rightharpoonup uw \quad \text{in } L^1_{loc}(\bar{\Omega} \times [0, \infty)), \quad (55)$$

with some limit functions u, v and w which are such that $u \geq 0, v \geq 0, w \geq 0$ a.e. in $\Omega \times (0, \infty)$.

Proof. Corollary 3.4 implies (47) and (48) hold, along a subsequence (49)-(51) can be achieved through a straightforward extraction procedure according to Lemmas 3.1 and 3.5. Similarly, (52)-(54) can be achieved in accordance with Lemmas 3.1 and 3.6. Similar to (7.7) [33, Lemma 7.1], (55) can be obtained. This completes the proof.

Lemma 3.7. Let $\alpha > \frac{N+1}{N+2}$ and assume that (u_0, v_0, w_0) satisfy (4). Furthermore, let u, v, w denote the limit function provided by Lemma 3.7. Then

$$\int_\Omega u(\cdot, t) \leq e^{-t} \int_\Omega u_0 + \kappa|\Omega|(1 - e^{-t}) \quad \text{for } a.e. \ t > 0, \quad (56)$$

and v and w fulfill the weak solution properties of (11) and (12), respectively.

Proof. From Lemma 2.2, we know that $\int_\Omega u_\varepsilon(\cdot, t) \leq e^{-t} \int_\Omega u_0 + \kappa|\Omega|(1 - e^{-t})$ for all $t > 0$ and each $\varepsilon \in (0,1)$, (47) and Fatou's lemma imply $u \in L^1_{loc}(\bar{\Omega} \times [0, \infty))$ as well as $\int_\Omega u(\cdot, t) \leq e^{-t} \int_\Omega u_0 + \kappa|\Omega|(1 - e^{-t})$ for a.e. $t > 0$. $v \in L^2_{loc}([0, \infty); W^{1,2}(\Omega))$ and $w \in L^2_{loc}([0, \infty); W^{1,2}(\Omega))$ are the directly consequences of (49), (51), (52) and (54). Multiplying the second equation in (14) by φ, we obtain

$$-\int_0^\infty \int_\Omega v_\varepsilon \varphi_t - \int_\Omega v_0 \varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla \varphi + \int_0^\infty \int_\Omega u_\varepsilon w_\varepsilon \varphi - \int_0^\infty \int_\Omega v_\varepsilon \varphi \quad (57)$$

for each $\varepsilon \in (0,1)$, by (49), (51) and (55) we have

$$-\int_0^\infty \int_\Omega v_\varepsilon \varphi_t \to -\int_0^\infty \int_\Omega v\varphi_t, \quad -\int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla \varphi \to -\int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi,$$

as well as

$$\int_0^\infty \int_\Omega u_\varepsilon w_\varepsilon \varphi \to \int_0^\infty \int_\Omega uw\varphi, \quad -\int_0^\infty \int_\Omega v_\varepsilon \varphi \to -\int_0^\infty \int_\Omega v\varphi$$

taking $\varepsilon \searrow 0$ in (57) yields (11). Similarly to (57), multiplying the third equation in (14) by φ, we have

$$-\int_0^\infty \int_\Omega w_\varepsilon \varphi_t - \int_\Omega w_0 \varphi(\cdot, 0) = -\int_0^\infty \int_\Omega \nabla w_\varepsilon \cdot \nabla \varphi + \int_0^\infty \int_\Omega v_\varepsilon \varphi - \int_0^\infty \int_\Omega w_\varepsilon \varphi \quad (58)$$

for each $\varepsilon \in (0,1)$, by (49), (52) and (54) we have

$$-\int_0^\infty \int_\Omega w_\varepsilon \varphi_t \to -\int_0^\infty \int_\Omega w\varphi_t, \quad -\int_0^\infty \int_\Omega \nabla w_\varepsilon \cdot \nabla \varphi \to -\int_0^\infty \int_\Omega \nabla w \cdot \nabla \varphi$$

as well as

$$\int_0^\infty \int_\Omega v_\varepsilon \varphi \to \int_0^\infty \int_\Omega v_\varepsilon \varphi, \quad -\int_0^\infty \int_\Omega w_\varepsilon \varphi \to -\int_0^\infty \int_\Omega w \varphi$$

taking $\varepsilon \searrow 0$ in (58) yields (12). This completes the proof.

Lemma 3.8. There exists a null set $\Lambda \subset (0, \infty)$ such that the limit functions u, v, w obtained in Lemma 3.7 fulfill the inequality

$$\frac{1}{2} \int_\Omega v^2(\cdot, T) - \frac{1}{2} \int_\Omega v_0^2 + \int_0^T \int_\Omega |\nabla v|^2 \geq -\int_0^T \int_\Omega v^2 + \int_0^T \int_\Omega uvw \tag{59}$$

for all $T \in (0, \infty) \setminus \Lambda$.

Proof. The proof similar to the recent work [22, Lemma 3.9], so we omit it.

Lemma 3.9. Let $\alpha > \frac{N+1}{N+2}$ and assume that (u_0, v_0, w_0) satisfy (4). Moreover, denote by $(\varepsilon_j)_{j \in \mathbb{N}}$ and u, v, w obtained in Lemma 3.7. Then there exist a subsequence $(\varepsilon_{j_k})_{k \in \mathbb{N}}$ and a null set $\Lambda \subset (0, \infty)$ such that $T \in (0, \infty) \setminus \Lambda$ the classical solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ of (14) fulfills

$$\nabla v_\varepsilon \to \nabla v \text{ in } L^2(\Omega \times (0, T)) \text{ as } \varepsilon = \varepsilon_j \searrow 0. \tag{60}$$

Proof. The proof similar to our recent works [22, Lemma 3.10] and [24, Lemma 3.11], so we omit it.

Lemma 3.10. Let $\alpha > \frac{N+1}{N+2}$. Suppose that (u_0, v_0, w_0) satisfy (4) and u, v, w obtained in Lemma 3.7. Furthermore, $\Phi(s) := \ln(s + 1)$ for $s \geq 0$. Then u is a global Φ-supersolution of (1) in the sense of Definition 2.1.

Proof. Using (48), it is readily known that $\Phi'(u)uw = \frac{uw}{u+1} \in L^1_{loc}(\bar\Omega \times [0, \infty))$, $\Phi(u) \in L^1_{loc}(\bar\Omega \times [0, \infty))$ and $u\Phi'(u) = \frac{u}{u+1} \in L^2_{loc}(\bar\Omega \times [0, \infty))$. Furthermore, (48) can be insure that

$$\Phi''(u)|\nabla u|^2 = -\frac{|\nabla u|^2}{(u+1)^2} = -|\nabla \ln(u+1)|^2 \in L^1_{loc}(\bar\Omega \times [0, \infty)),$$

and since

$$|u\Phi''(u)\nabla u| = \frac{u|\nabla u|}{(u+1)^2} \leq |\nabla \ln(u+1)|,$$

we have $u\Phi''(u)\nabla u \in L^2_{loc}(\bar\Omega \times [0, \infty))$. We fix an arbitrary nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0, \infty))$ with $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega \times (0, \infty)$, and then multiply the first equation in (14) by $\frac{1}{u_\varepsilon + 1}\varphi$ and fix $T > 0$ such that $\varphi \equiv 0$ in $\Omega \times (T, \infty)$, and integrate by parts, we obtain

$$\begin{aligned}
\int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2}\varphi = &-\int_0^\infty \int_\Omega \ln(u_\varepsilon + 1)\varphi_t - \int_0^\infty \int_\Omega \ln(u_0 + 1)\varphi(\cdot, 0) \\
&+ \int_0^\infty \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)^2} \nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \varphi \\
&- \int_0^\infty \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}(S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \nabla\varphi \\
&- \int_0^\infty \int_\Omega \ln(u_\varepsilon + 1)\Delta\varphi - \int_0^\infty \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}w_\varepsilon \cdot \varphi \\
&+ \int_0^\infty \int_\Omega \kappa \int_\Omega \frac{1}{u_\varepsilon + 1}\varphi - \int_0^\infty \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}\varphi
\end{aligned} \tag{61}$$

for all $\varepsilon \in (0, 1)$. Using (48) we have

$$-\int_0^\infty \int_\Omega \ln(u_\varepsilon + 1)\varphi_t \to -\int_0^\infty \int_\Omega \ln(u + 1)\varphi_t \tag{62}$$

as well as

$$-\int_0^\infty \int_\Omega \ln(u_\varepsilon + 1)\Delta\varphi \to -\int_0^\infty \int_\Omega \ln(u + 1)\Delta\varphi, \tag{63}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, by the definition of S_ε and (3), we have

$$\left|\frac{u_\varepsilon}{u_\varepsilon + 1}S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\right| \leq S_0 \text{ in } \Omega \times (0, \infty) \text{ for all } \varepsilon \in (0, 1)$$

and (47) and (50) also imply

$$\frac{u_\varepsilon}{u_\varepsilon + 1}S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \to \frac{u}{u + 1}S(x, u, v), \tag{64}$$

by Lemma 3.10, we have

$$\nabla v_\varepsilon \to \nabla v \ \text{ in } L^2_{loc}(\overline{\Omega} \times [0, \infty)) \tag{65}$$

and implies that

$$\frac{u_\varepsilon}{u_\varepsilon + 1}(S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \to \frac{u}{u + 1}(S(x, u, v)\nabla v) \ \text{ in } L^2_{loc}(\overline{\Omega} \times [0, \infty)), \tag{66}$$

as $\varepsilon = \varepsilon_j \searrow 0$, which directly implies

$$-\int_0^\infty \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1}(S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \nabla\varphi \to -\int_0^\infty \int_\Omega \frac{u}{u + 1}(S(x, u, v)\nabla v) \cdot \nabla\varphi \tag{67}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Furthermore, combining with (48) and (66), we have

$$\int_0^\infty \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon + 1)^2}\nabla u_\varepsilon \cdot (S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon)\varphi$$
$$= \int_0^\infty \int_\Omega \nabla \ln(u_\varepsilon + 1) \cdot \left(\frac{u_\varepsilon}{u_\varepsilon + 1}S_\varepsilon(x, u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon\right) \cdot \varphi \to \int_0^\infty \int_\Omega \nabla \ln(u + 1) \cdot \left(\frac{u}{u + 1}S(x, u, v)\nabla v\right) \cdot \varphi \tag{68}$$
$$= \int_0^\infty \int_\Omega \frac{u}{(u + 1)^2}\nabla u \cdot (S(x, u, v)\nabla v) \cdot \varphi$$

Combining with (62), (63), (67) and (68), we infer from (61) that

$$\int_0^\infty \int_\Omega \frac{|\nabla u|^2}{(u + 1)^2}\varphi \le \liminf_{\varepsilon = \varepsilon_j \searrow 0} \int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2}\varphi$$
$$= -\int_0^\infty \int_\Omega \ln(u + 1)\varphi_t - \int_0^\infty \int_\Omega \ln(u_0 + 1)\varphi(\cdot, 0)$$
$$+ \int_0^\infty \int_\Omega \frac{u}{(u + 1)^2}\nabla u \cdot (S(x, u, v)\nabla v) \cdot \varphi$$
$$- \int_0^\infty \int_\Omega \frac{u}{u + 1}(S(x, u, v)\nabla v) \cdot \nabla\varphi \tag{69}$$
$$- \int_0^\infty \int_\Omega \ln(u + 1)\Delta\varphi - \int_0^\infty \int_\Omega \frac{u}{u + 1}w \cdot \varphi$$
$$+ \int_0^\infty \int_\Omega \kappa \int_\Omega \frac{1}{u + 1}\varphi - \int_0^\infty \int_\Omega \frac{u}{u + 1}\varphi,$$

which means that u is a global weak Φ-supersoluton of (9).

Finally, we prove the main theorem.

The proof of Theorem 2.1 Combining with Lemma 3.8 and 3.11, the desired result is obtained.

4. Application to Optimal Control

In this section, to apply the existence results to prove the existence of the optimal control pair, we rewrite system (1), (4) and (5) into the following form and give the discussion of optimal control problem.

We are concerned with optimal control problem

$$Min \ \frac{1}{2}\int_0^T \int_\Omega |\mathfrak{B}(p(t, x) - p_0(t, x))|^2 dxdt + \int_0^T h(U(t))dt \quad (P) \tag{70}$$

subject to

$$\begin{cases} p'(t) + Ap(t) + B(p(t)) = U(t) + f_0(t) & \text{in } \Omega \times (0, T), \\ p(0) = p_0 & \text{in } \Omega, \end{cases} \tag{71}$$

with state constraint

$$p(t) \in \mathcal{K} \quad \forall t \in [0, T],$$

where $\mathcal{K}$ is a close convex subset in

$$\mathcal{H} = \{p, \ p \in (L^2(0,T;\mathcal{W}))^3, \ \frac{\partial p}{\partial \nu} = 0 \text{ on } \partial\Omega\}.$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$ is a given constant, $\mathcal{W}$ is a Hilbert space, and $f_0(t) \in L^2(0,T;(L^2(\Omega))^3)$. The function $h : \mathcal{W} \to (-\infty, +\infty]$ is convex and lower semicontinuous function. Moreover, there exist $\alpha > 0$ and $C \in \mathbb{R}$ such that

$$h(U) \geq \alpha |U|_{\mathcal{W}}^2 + C, \quad \forall p \in \mathcal{W}. \tag{72}$$

$p_0 \in L^2(0,T;\mathcal{H})$ and $\mathfrak{B} \in L(\mathcal{V},\mathcal{H})$, where $\mathcal{V} = (H_0^1(\Omega))^3 \cap \mathcal{H}$, here $\mathfrak{B} := l_1 I$. Denote by the symbol $\|\cdot\|$ the norm of the space $\mathcal{V}$, which is defined by

$$\|p\|^2 = \sum_{i=1}^3 \int_\Omega |\nabla p_i|^2$$

and by the symbol $|\cdot|$ the norm of $\mathbb{R}^3$ and $(L^2(\Omega))^3$. We endow the space $\mathcal{H}$ with the norm of $(L^2(\Omega))^3$, and presented by $\langle \cdot, \cdot \rangle$ the scalar product of $\mathcal{H}$, $\langle \cdot, \cdot \rangle_{(\mathcal{V},\mathcal{V}')}$ the paring between $\mathcal{V}$ and its dual $\mathcal{V}'$ with the norm $\|\cdot\|_{\mathcal{V}'}$. Let $A \in L(\mathcal{V},\mathcal{V}')$ and trilinear function be defined by:

$$\langle Ap, y \rangle = \sum_{i=1}^3 \int_\Omega \nabla p_i \cdot \nabla y_i, \quad \forall p, y \in \mathcal{V}$$

and

$$b(p, y, z) = \sum_{i=1}^3 \int_\Omega p_i D_i y_j D_i z_j$$

respectively, where $D_i = \frac{\partial}{\partial x_i}$, $D(A) = (H^2(\Omega))^3 \cap \mathcal{V}$. If there is no confuse, we present also by $\langle \cdot, \cdot \rangle$ the dual product between $\mathcal{V}$ and its dual $\mathcal{V}'$. We define the operators $B : \mathcal{V} \to \mathcal{V}'$ by

$$\langle B(p), y \rangle = b(p, p, y) \quad \forall y \in \mathcal{V}.$$

Let $f(t) = Pf_0(t)$ and $D \in L(\mathcal{W},\mathcal{H})$ is given by $D = PI$, where $I \in L(\mathcal{W};(L^2(\Omega))^3)$ is a unit matrix, $P : (L^2(\Omega))^3 \to \mathcal{H}$ is the projection on $\mathcal{H}$. Then we may rewrite the optimal control problem (P) as

$$Min \ \frac{1}{2}\int_0^T |\mathfrak{B}(p(t) - p_0(t))|^2 dxdt + \int_0^T h(U(t))dt \quad (P) \tag{73}$$

subject to

$$\begin{cases} p'(t) + Ap(t) + B(p(t)) = DU(t) + f(t) & \text{in } \Omega \times (0,T), \\ p(0) = p_0 & \text{in } \Omega, \end{cases} \tag{74}$$

with state constraint

$$p(t) \in \mathcal{K} \quad \forall t \in [0,T],$$

where

$$A = \begin{pmatrix} -\Delta & 0 & 0 \\ 0 & -\Delta & 0 \\ 0 & 0 & -\Delta \end{pmatrix},$$

$U(t) = (0,0,\lambda(t,x))^T$, $p(t) = (u,v,w)^T$, $B(p(t)) = \nabla \cdot (\frac{u}{(1+u)^\alpha}\nabla v)$, $f(t) = (f_1(t), f_2(t), f_3(t))^T$ with

$$\begin{cases} f_1(t) = -uw + \kappa - u, \\ f_2(t) = uw - v, \\ f_3(t) = v - w, \end{cases}$$

$f, DU \in L^2(0,T;\mathcal{H})$. Let $\mathcal{U} = L^2(0,T;\mathcal{W})$ and

$$\mathcal{U}_{ad} = \left\{ \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \in \mathcal{U}; \ \lambda \in L^2(0,T;\mathcal{W}), \ \lambda \geq 0, \ \|D\lambda\|_{L^2(0,T;\mathcal{W})} \leq C \right\},$$

where $\lambda(t,x)$ denote control variable. We assume $\mathcal{U}_{ad}$ is a closed, bounded, and convex subset of $\mathcal{U}$.

Then, we have the following existence theorem.

Lemma 4.1. The optimal control problem (P) has at least one optimal pair $(\bar{p}, \bar{U})$.

Proof. We denote

$$J(p,U) = \frac{1}{2}\int_0^T |\mathfrak{B}(p(t) - p_0(t))|^2 dxdt + \int_0^T h(U(t))dt,$$

$$\gamma = \inf\left\{\frac{1}{2}\int_0^T |\mathfrak{B}(p(t) - p_0(t))|^2 dxdt + \int_0^T h(U(t))dt; (p,U) \in \mathcal{F}_w\right\}.$$

Then there exist $(p_n, U_n) \in \mathcal{F}_w$ such that

$$\gamma \leq J(p_n, U_n) \leq \gamma + \frac{1}{N}. \tag{75}$$

By (72) and (75), it follow that $\{U_n\}$ is bounded in $L^2(0,T;\mathcal{W})$. Thus, there exists at least one subsequence which denoted again by $\{U_n\}$, such that

$$U_n \rightharpoonup \bar{U} \quad \text{in } L^2(0,T;\mathcal{W}). \tag{76}$$

Multiplying equation

$$\begin{cases} p_n'(t) + Ap_n(t) + B(p_n(t)) = DU_n(t) + f(t) & \text{in } \Omega \times (0,T), \\ p(0) = p_0 & \text{in } \Omega, \end{cases} \tag{77}$$

by p_n, integrating on $(0,t)$, we obtain that

$$|p_n(t)|^2 + \int_0^t \|p_n\|^2 ds \leq c_1 + c_2 \int_0^t |p_n(s)|^2 ds,$$

by Gronwall's inequality, we have

$$|p_n(s)|^2 + \int_0^T \|p_n\|^2 dt \leq C. \tag{78}$$

This yields that

$$p_n \rightarrow \bar{p} \quad \text{weakly}^* \text{ in } L^\infty(0,T;\mathcal{H}), \tag{79}$$

$$p_n \rightharpoonup \bar{p} \quad \text{in } L^2(0,T;\mathcal{V}), \tag{80}$$

$$Ap_n \rightharpoonup A\bar{p} \quad \text{in } L^2(0,T;\mathcal{V}'). \tag{81}$$

By the properties of the trilinear function b, we derive that

$$|\langle Bp_n, y\rangle_{(\mathcal{V}',\mathcal{V})}| \leq c_3 |p_n|^{\frac{1}{2}} \|p_n\|_{L^1(\Omega)}^{\frac{3}{2}} \|y\|$$

and it follows that

$$\int_0^T |Bp_n|_{\mathcal{V}'}^{\frac{4}{3}} dt \leq c_4 \int_0^T \|y\|^2 dt \leq c_5. \tag{82}$$

Therefore,

$$\int_0^T |\frac{dp_n}{dt}|_{\mathcal{V}'}^{\frac{4}{3}} dt \leq c_6, \tag{83}$$

from (82) and (83), we have

$$\frac{dp_n}{dt} \rightharpoonup \frac{d\bar{p}}{dt} \quad \text{in } L^{\frac{4}{3}}(0,T;\mathcal{V}'), \tag{84}$$

$$Bp_n \rightharpoonup \vartheta \quad \text{in } L^{\frac{4}{3}}(0,T;\mathcal{V}'). \tag{85}$$

To reveal that $(\bar{p}, \bar{U})$ fulfills (75), it remains to prove that $\vartheta = B\bar{p}$ a.e. in $(0,T)$. By (79)-(81), (83) and Aubin's

compactness theorem [2, Theorem 1.20], we have

$$\bar{p}_n \rightarrow \bar{p} \quad \text{strongly in } L^2(0,T;\mathcal{H}), \tag{86}$$

and

$$\int_0^T |\langle Bp_n - B\bar{p}, \varphi\rangle_{(\mathcal{V}',V)}| \leq \int_0^T (|b(p_n - \bar{p}, p_n, \varphi)| + |b(\bar{p}, p_n - \bar{p}\varphi)|)dt \rightarrow 0 \tag{87}$$

as $n \rightarrow \infty$, $\forall \varphi \in L^2(0,T;C_0^\infty(\Omega))$. Thus, $\delta(t) = B(\bar{p}(t))$ a.e. in $(0,T)$. Since h is convex and lower semicontinuous, we obtain that

$$\gamma \leq J(\bar{p}, \bar{U}) \leq \liminf_{n\rightarrow\infty} J(p_n, U_n) \leq \gamma, \tag{88}$$

we also have that for each $t \in [0,T]$, $\exists t_n \in (0,T)$ such that $\bar{p}(t_n) \in \mathcal{K}$, and

$$\bar{p}(t_n) \rightharpoonup \bar{p}(t) \quad \text{in } \mathcal{H}.$$

Since $\mathcal{K}$ is close convex subset of $\mathcal{H}$, it is weakly closed, it is follows that $\bar{p}(t) \in \mathcal{K}$, $\forall t \in [0,T]$. Therefore, $(\bar{p}, \bar{U})$ is an optimal pair for problem (P). This completes the proof.

5. Conclusions

In this paper, we mainly investigate a virus infection model with saturated chemotaxis. Our result in Theorem 1, as $N = 1, 2$, which is consistent with the result Theorem 1.1 [27], as $N \geq 3$, Theorem 1 generalizes and improves the result Theorem 1.1 [27]. And moreover we first prove that the optimal control problem (P) has at least one optimal pair

$(\bar{p}, \bar{U})$.

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